Solving Trig Equations, Part III

It's time to add the last few tricks to our trig-equation-solving toolbox. The first is to learn how to solve equations that have more than just an x (or t or θ or whatever) inside of the trig function. For example, how would we solve something like $\sin(2x) = \frac{\sqrt{3}}{2}$?

Let's use that as an example. If we want to solve $\sin(2x) = \frac{\sqrt{3}}{2}$, we can first substitute something for $2x$, say, $z = 2x$. Solving $\sin(z) = \frac{\sqrt{3}}{2}$ is easy enough, z must be $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \ldots$. If $2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \ldots$, then $x = (\frac{\pi}{3})/2, (\frac{2\pi}{3})/2, (\frac{7\pi}{3})/2, (\frac{8\pi}{3})/2, \ldots$, or in other words, $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}, \ldots$.

Notice that if we wanted to restrict to values of x in the interval $[0, 2\pi)$, we would have to take values of z in the interval $[0, 4\pi)$, since $0 \le x < 2\pi$ gives $0 \le 2x < 4\pi$.

Find all x in $[0, 2\pi)$ satisfying the following equations.

(1) $\cos(2x) = -\frac{\sqrt{2}}{2}$

(2) $\sec(3x) = -1$

(3) $\sin(\frac{x}{2}) = \frac{\sqrt{3}}{2}$

(4) $\tan(2x) = \sqrt{3}$

The next trick we'll use for solving trig equations is the use of trigonometric identities. Trigonometric identities are equations that are always true. For example, the identity $\sin^2 x + \cos^2 x = 1$ is true, *no matter what x is.* You can see this by looking at the unit circle. Pick an angle, say, $\frac{\pi}{3}$, and draw the associated triangle by connecting the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ down to the x-axis. The sine represents the height of this triangle, and the cosine represents the length of the base. By the Pythagorean theorem, the square of the height plus the square of the base equals the square of the hypotenuse. Since this is the *unit* circle, and the hypotenuse is the length of the radius, the hypotenuse equals 1. Therefore, $\sin^2 x + \cos^2 x = 1$ for any value of x. There are many more trigonometric identities. You don't need to memorize them, but you should know how to use them. First, we'll list the major ones for your reference, and then go through an example.

Trigonometric Identities

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \csc^2 x$
- $\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_1 \sin x_2$
- $\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$
- $\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$
- $\sin x_1 \cos x_2 = \frac{1}{2}(\sin(x_1 - x_2) + \sin(x_1 + x_2))$
- $\cos x_1 \cos x_2 = \frac{1}{2}(\cos(x_1 - x_2) + \cos(x_1 + x_2))$
- $\sin x_1 \sin x_2 = \frac{1}{2}(\cos(x_1 - x_2) - \cos(x_1 + x_2))$

Now for an example. Suppose we want to solve the equation

$$\sec^2 x - 2\tan x = 4.$$

It's hard to deal with multiple trig functions at once, and there's no way to factor this equation, so we must use a trig identity first. We note that there's an identity involving secant and tangent, so let's use that one, and replace $\sec^2 x$ by $\tan^2 x + 1$:

$$\tan^2 x + 1 - 2\tan x = 4$$

Now this has a chance of being factored. If we get all of the terms to one side, we have

$$\tan^2 x - 2\tan x - 3 = 0$$

which simplifies to

$$(\tan x - 3)(\tan x + 1) = 0$$

so $\tan x = 3$ or $\tan x = -1$.

Here, we get a chance to use our final "trick." The equation $\tan x = -1$ is easy to solve via the unit circle. If $\tan x = -1$, then x must be $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$ or one of these angles plus some multiple of 2π. However, look through all of your memorized angles in the unit circle and you won't find one where the tangent is 3. That's okay, though – we can write our solution in terms of the arctangent function. If $\tan x = 3$, then one possible value for x is $x = \arctan 3$ (or $\tan^{-1} 3$). The value $\arctan 3$ would be a value for x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, since that is the range of the inverse tangent function. (Furthermore, it really should be in the interval $[0, \frac{\pi}{2})$ since that's where the tangent function is positive.) There are other angles that have a tangent of 3, however – finding them may take a bit of detective work. The angle $\arctan 3$ sits somewhere in the first quadrant, at a spot (x, y) where $\frac{x}{y} = 3$. There must be a corresponding spot in the third quadrant, at $(-x, -y)$, where $\frac{-x}{-y} = 3$ (since the negatives would just cancel). The relationship between the point (x, y) and the point $(-x, -y)$ is that they are on opposite ends of the unit circle (try drawing it out if you don't believe this). In other words, they are π radians apart. Therefore, another angle x with $\tan x = 3$ is $\arctan 3 + \pi$. And then, of course, we have those two angles, $\arctan 3$ and $\arctan 3 + \pi$, plus all multiples of 2π. So, our final solution set to this equation is:

$$x = \left\{ \frac{3\pi}{4}, \frac{7\pi}{4}, \arctan 3, \text{ or } (\arctan 3 + \pi) \right\} + 2k\pi \text{ for some integer } k.$$

For the following exercises, find all solutions x in $[0, 2\pi)$.

(5) $\sin(2x) = \cos x$

(6) $2\cot^2 x = 3\csc x$

(7) $\cos(4x) = \cos(2x)$

(8) $2\cos^2\left(\frac{x}{2}\right) + \sin^2 x = 0$